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A dual form for discretized kinematic formulation in shakedown analysis

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Abstract

Shakedown analysis with the discretization in kinematically admissible finite elements and the use of von Mises yield criterion is considered in this paper. The shakedown load multiplier formulated by kinematic theorem under nonlinear form is proved to be the primal form of the shakedown load multiplier formulated by static theorem. Based on this duality, an efficient dual algorithm for shakedown analysis of structures is established and implemented connecting with finite element discretization technique. Some numerical examples are presented.

Keywords: Shakedown analysis; Limit analysis; Duality; Nonlinear optimization; Direct limit state analysis

1. Introduction

From both physical and mathematical point of views, the duality that connects the upper and lower bound theorems is an attractive property of shakedown theory. The problem of duality in shakedown has been addressed in literature since longtime.

Maier (1973) showed that by using mathematical programming method, the precise duality of static and kinematic approaches could be obtained. Studies on duality were also carried out by De Saxcé (1986), Morelle (1989), Kamenjarzh and Weichert (1992), etc. Kamenjarzh and Weichert used a convex-analysis approach to establish a dual kinematic safety factor based on Melan's theorem. The duality was restricted to the case of spherical yield surfaces that can be found when applying von Mises yield condition to some thin-walled structures. The dual theorem was then generalized in Kamenjarzh and Merzljakov (1994a) to cover a wider class of yield surfaces such as cylindrical yield surfaces as well as those with different yield stresses at tension and compression. The theory was later modified to give an explicit kinematic formulation in Kamenjarzh and Merzljakov (1994b). In their work, duality conditions were presented and a discretized version of the dual function was considered by mean of finite elements. By considering the plastic shakedown analysis, Polizzotto (1993) also showed that dualization between static and kinematic formulations is

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straightforward when the set of variable loads is a polyhedron. Extended theory of duality for convex nonpolyhedral load domain can be found in the work of Silveira and Zouain (1997). Using the kinematic formulation for incremental collapse, they obtained a static dual form. Further studies concerning dual bounds of shakedown loads were carried out by Zouain and Silveira (1999).

However, the duality between the discretized forms of upper and lower formulations by using kinematically admissible finite elements and von Mises criterion has not yet been established, despite the fact that such dual forms are widely used in practical optimization computations. On the other hand, the duality in discretized forms has never been practically employed in shakedown analysis to improve numerical calculations.

In this work, both upper and lower bound formulations of the shakedown load multiplier are discretized by kinematically admissible finite elements, and von Mises yield criterion are adopted. The two discretized forms are shown to be primal and dual, respectively. Based on this duality, an efficient algorithm for shakedown analysis is constructed. As will be shown hereafter, by using an appropriate finite element mesh the shakedown load calculation in this work leads to accurate solutions without strict bounding property.

2. Shakedown analysis as a problem of nonlinear programming

2.1. Kinematic formulation

Consider a structure made of elastic–perfectly plastic material and subjected to n time-dependent loads $\overline{P}_k^0(t)$, each of these loads may vary independently within a given range:

$$\overline{P}_k^0(t) \in I_k^0 = [\overline{P}_k^-, \overline{P}_k^+] = [\mu_k^-, \mu_k^+]P_k^0, \quad k = \overline{1, n} \tag{2.1}$$

where μ_k^-, μ_k^+ are, respectively, the lower and upper bounds of kth nominal load P_k^0. They form a convex polyhedral domain D of n dimensions with 2^n vertices in load space. This load domain can be represented in the following linear form (Konig, 1987):

$$P(t) = \sum_{k=1}^{n} \mu_k(t) P_k^0 \tag{2.2}$$

where

$$\mu_k^- \leqslant \mu_k(t) \leqslant \mu_k^+, \quad k = \overline{1, n} \tag{2.3}$$

In many cases, it is useful to describe this load domain in stress space. To this end, we use here the notion of a fictitious elastic response of the structure under the same loading. The fictitious elastic stress $\sigma_{ij}^{\mathrm{E}}(\mathbf{x}, t)$ is written in a form similar to (2.2):

$$\sigma_{ij}^{\mathrm{E}}(\mathbf{x}, t) = \sum_{k=1}^{n} \mu_k(t) \sigma_{ij}^{\mathrm{E}k}(\mathbf{x}, P_k^0) \tag{2.4}$$

where $\sigma_{ij}^{\mathrm{E}k}(\mathbf{x}, P_k^0)$ denotes the stress field of the structure when subjected to kth nominal load P_k^0.

Based on the kinematic theorem of Koiter, the upper bound of shakedown load multiplier may be formulated in the following form:

$$
\alpha^+ = \min \sum_{k\in I_D} \int_V D^{\mathrm{p}}(\dot{\varepsilon}_{ij}^k)\,\mathrm{d}V \qquad \text{(a)}
$$
$$
\text{s.t.} \begin{cases} \Delta\varepsilon_{ij} = \sum\limits_{k\in I_D} \dot{\varepsilon}_{ij}^k & \text{(b)} \\ \Delta\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \Delta u_i}{\partial x_j} + \frac{\partial \Delta u_j}{\partial x_i}\right) \quad \text{in } V & \text{(c)} \\ \Delta u_i = 0 \quad \text{on } A_u & \text{(d)} \\ \sum\limits_{k\in I_D} \int_V \sigma_{ij}^{\mathrm{E}}(\mathbf{x}, P_k^0)\dot{\varepsilon}_{ij}^k\,\mathrm{d}V = 1 & \text{(e)} \end{cases} \tag{2.5}
$$

where α^+ denotes the upper bound of the shakedown load multiplier; u is the displacement; $\dot{\varepsilon}_{ij}^k$ is the corresponding strain rate at load vertex k; $\Delta\varepsilon_{ij}$ is the total plastic strain increment after a loading cycle; I_D is the set of all load vertices and $D^{\mathrm{p}}(\dot{\varepsilon}_{ij}^k)$ denotes the plastic dissipation rate. By using von Mises yield criterion we have:

$$
D^{\mathrm{p}} = \sigma_p \left(\frac{3}{2}\dot{e}_{ij}^{\mathrm{p}}\dot{e}_{ij}^{\mathrm{p}}\right)^{1/2} \tag{2.6}
$$

where $\dot{e}_{ij}^{\mathrm{p}}$ denotes the deviatoric part of $\dot{\varepsilon}_{ij}^{\mathrm{p}}$, and σ_p is the yield stress of material.

The discretized form of (2.5) by means of the finite element method can be expressed as follows:

$$
\alpha^+ = \min \sum_{k=1}^{m} \sum_{i=1}^{\mathrm{NG}} \sqrt{2} w_i k_v \sqrt{\dot{\boldsymbol{\varepsilon}}_{ik}^{\mathrm{T}} \mathbf{D} \dot{\boldsymbol{\varepsilon}}_{ik} + \varepsilon_0^2} \qquad \text{(a)}
$$
$$
\text{s.t.} \begin{cases} \sum\limits_{k=1}^{m} \dot{\boldsymbol{\varepsilon}}_{ik} = \mathbf{B}_i \mathbf{q} \quad \forall i = \overline{1, \mathrm{NG}} & \text{(b)} \\ \mathbf{D}_v \dot{\boldsymbol{\varepsilon}}_{ik} = \mathbf{0} \quad \forall i = \overline{1, \mathrm{NG}} & \text{(c)} \\ \sum\limits_{k=1}^{m}\sum\limits_{i=1}^{\mathrm{NG}} w_i \dot{\boldsymbol{\varepsilon}}_{ik}^{\mathrm{T}} \boldsymbol{\sigma}_{ik}^{\mathrm{E}} = 1 & \text{(d)} \end{cases} \tag{2.7}
$$

where $k_v = \sigma_p/\sqrt{3}$; $\dot{\boldsymbol{\varepsilon}}_{ik}$ and $\boldsymbol{\sigma}_{ik}^{\mathrm{E}}$ denote, respectively, the vector of deformation rate and the vector of the fictitious elastic stress at Gauss point i and load vertex k; $\mathbf{q}$ is the nodal displacement vector; $\mathbf{B}_i$ is the strain matrix; $m = 2^n$; NG denotes the total number of Gauss points of the whole structure with integration weight w_i at Gauss point i; ε_0 is a small value of regularization.

We note that the shakedown load multiplier α^+ in Eq. (2.7) is an approximation of α^+ in Eq. (2.5). In order to obtain α^+ as an upper bound of the shakedown load multiplier, the following conditions must be fulfilled:

1. The number of Gauss points must be sufficient so that: the volume integrations (2.7.a) and (2.7.d) are calculated precisely, the numbers of constraints in (2.7.b) and (2.7.c) are sufficient to ensure compatibility and incompressibility over each element.
2. The fictitious elastic stresses in (2.7) are computed exactly.

Although we may fulfill the first requirement, the fictitious elastic stresses are normally approximations of the exact values. Therefore, the shakedown load multiplier α^+ computed in (2.7) may lose its bounding characteristic in the strict sense.

2.2. Static formulation

Based on the static theorem of Melan, the lower bound of shakedown load multiplier can be written as:

$$\alpha^- = \max \alpha \quad \text{(a)}$$
$$\text{s.t.} \begin{cases} \partial_j \bar{\rho}_{ij} = 0 & \text{in } V & \text{(b)} \\ \bar{n}_j \bar{\rho}_{ij} = 0 & \text{on } A_\sigma & \text{(c)} \\ f(\alpha \sigma^{\mathrm{E}}_{ij}(\mathbf{x}, \widehat{P}_k) + \bar{\rho}_{ij}) \leqslant 0 & \forall k = \overline{1,m} & \text{(d)} \end{cases} \tag{2.8}$$

Evidently it is reasonable to use statically admissible elements to discretize (2.8). However, as we know, those elements are not widely used. By using kinematically admissible elements instead, the lower bound of shakedown load multiplier may be discretized as follows:

$$\alpha^- = \max \alpha \quad \text{(a)}$$
$$\text{s.t.} \begin{cases} \sum_{i=1}^{\mathrm{NG}} w_i \mathbf{B}_i^{\mathrm{T}} \bar{\boldsymbol{\rho}}_i = \mathbf{B}^{\mathrm{T}} \bar{\boldsymbol{\rho}} = \mathbf{0} & & \text{(b)} \\ f(\alpha \boldsymbol{\sigma}^{\mathrm{E}}_{ik} + \bar{\boldsymbol{\rho}}_i) \leqslant 0 & \forall k = \overline{1,m},\ \forall i = \overline{1,\mathrm{NG}} & \text{(c)} \end{cases} \tag{2.9}$$

The discretized fictitious elastic stress field $\boldsymbol{\sigma}^{\mathrm{E}}_{ik}$ in this formulation is computed at Gauss points like the residual stress field $\bar{\boldsymbol{\rho}}_i$. Note that the nonlinear mathematical optimization problem (2.9) has $\mathrm{NSC} \times \mathrm{NG} + 1$ variables: the global residual stress $\bar{\boldsymbol{\rho}}$ and load multiplier α. Here NSC is the number of stress components.

Again, we note that the shakedown load multiplier α^- of the formulation (2.9) is an approximation of α^- in formulation (2.8). In order to obtain α^- as a lower bound of the shakedown load multiplier, the following conditions must be fulfilled (Stein et al., 1993):

1. The static equilibrium condition (2.8.b) and the static boundary condition (2.8.c) for the residual stress distribution $\bar{\rho}_{ij}$ are satisfied exactly for all points x in the volume V or on the surface A_σ.
2. The fictitious elastic stress field $\sigma^{\mathrm{E}}_{ij}(\mathbf{x}, \widehat{P}_k)$ is calculated exactly for all points $\mathbf{x}$ in V and for all load vertices $\widehat{P}_k$ of the given load domain.
3. The yield condition (2.8.d) is satisfied exactly for all points x in V.

Unfortunately we can only have a quasi-equilibrium state (condition (2.9.b)) as well as yield criterion satisfied at discrete Gauss points of the structure (condition (2.9.c)). Besides, as already noted above, the fictitious elastic stresses are normally approximations of the exact values. Therefore the shakedown load multiplier α^- obtained by (2.9) has no strict bounding characteristic.

If the von Mises yield criterion is used, the requirement (2.9.c) represents a system of NG nonlinear inequality constraints. These nonlinear constraints are the major obstacle in implementing the von Mises yield criterion in shakedown analysis. To overcome this difficulty, Tresca or linearized von Mises yield criterion may be used. In order to avoid the use of linearization, Stein et al. (1993) proposed a sequential quadratic procedure with a basic-reduction technique. This technique permits us to consider only some small number of residual stress vectors as active variables at a time. Based on this set of active variables, the global residual stress vector is improved, a new set of active variables is sought for and the procedure continues until a solution is found.

We can also overcome the difficulty stemming from the nonlinear constraints of (2.9) if we do not directly maximize the load multiplier, but consider it as a complementary problem of the kinematic formulation

(2.7). In order to do so, the duality between formulations (2.7) and (2.9) will be discussed in details in the following section.

3. The duality between kinematic and static formulations

By restricting ourselves to polyhedral load domains, we show in this section that the static formulation (2.9) may be viewed as the dual form of its kinematic counterpart (2.7). For the sake of simplicity, let us rewrite the formulation (2.7) in a simpler form by setting:

- The new strain rate vector $\mathbf{e}_{ik}$ (the dot mark denoting time derivative is omitted for simplicity):

$$\mathbf{e}_{ik} = w_i \mathbf{D}^{1/2} \dot{\boldsymbol{\varepsilon}}_{ik} \tag{3.1}$$

- The new fictitious elastic stress field $\mathbf{t}_{ik}$:

$$\mathbf{t}_{ik} = \mathbf{D}^{-1/2} \boldsymbol{\sigma}_{ik}^{\mathrm{E}} \tag{3.2}$$

- The new deformation matrix $\widehat{\mathbf{B}}_i$:

$$\widehat{\mathbf{B}}_i = w_i \mathbf{D}^{1/2} \mathbf{B}_i \tag{3.3}$$

In the above definition $\mathbf{D}^{1/2}$ and $\mathbf{D}^{-1/2}$ are a symmetric matrices such that:

$$\mathbf{D}^{-1/2} = (\mathbf{D}^{1/2})^{-1} \quad \text{and} \quad \mathbf{D} = \mathbf{D}^{1/2}\mathbf{D}^{1/2} \tag{3.4}$$

With these definitions, the objective function of (2.7) becomes:

$$\sum_{k=1}^{m} \sum_{i=1}^{\mathrm{NG}} \sqrt{2} w_i k_v \sqrt{\dot{\boldsymbol{\varepsilon}}_{ik}^{\mathrm{T}} \mathbf{D} \dot{\boldsymbol{\varepsilon}}_{ik} + \varepsilon_0^2} = \sqrt{2} k_v \sum_{k=1}^{m} \sum_{i=1}^{\mathrm{NG}} \sqrt{\mathbf{e}_{ik}^{\mathrm{T}} \mathbf{e}_{ik} + \varepsilon^2} \tag{3.5}$$

In formulation (3.5), ε^2 is a small positive number that is suitably chosen to avoid the singularity of the objective function. By substituting (3.1)–(3.5) into (2.7) one obtains a simplified formulation:

$$\begin{aligned}
&\alpha^{+} = \min \sum_{i=1}^{\mathrm{NG}} \sum_{k=1}^{m} \sqrt{2} k_v \sqrt{\mathbf{e}_{ik}^{\mathrm{T}} \mathbf{e}_{ik} + \varepsilon^2} && \text{(a)} \\
&\text{s.t.} \begin{cases}
\sum_{k=1}^{m} \mathbf{e}_{ik} - \widehat{\mathbf{B}}_i \mathbf{q} = \mathbf{0} & \forall i = \overline{1, \mathrm{NG}} \quad \text{(b)} \\
\frac{1}{3} \mathbf{D}_v \mathbf{e}_{ik} = \mathbf{0} & \forall i = \overline{1, \mathrm{NG}}, \ \forall k = \overline{1, m} \quad \text{(c)} \\
\sum_{i=1}^{\mathrm{NG}} \sum_{k=1}^{m} \mathbf{e}_{ik}^{\mathrm{T}} \mathbf{t}_{ik} - 1 = \mathbf{0} & \text{(d)}
\end{cases}
\end{aligned} \tag{3.6}$$

where factor (1/3) is added in (3.6.c) for a technical reason.

As it is well known, limit analysis may be considered as a special case of shakedown analysis. Andersen et al. (2000), while considering a problem of minimizing a sum of Euclidean norms, found that in the case of limit analysis there exists a dual form for (3.6). A generalization of this dual form is presented hereafter through the following propositions:

Proposition 1. *If there exists a finite solution α^+ for the kinematic shakedown load multiplier* (3.6) *with* $\varepsilon^2 = 0$*, then* α^+ *has its dual form as:*

$$\alpha^- = \max_{\boldsymbol{\gamma}_{ik},\boldsymbol{\beta}_i,\alpha} \alpha \quad \text{s.t.} \quad \begin{cases} \|\boldsymbol{\gamma}_{ik} + \boldsymbol{\beta}_i + \mathbf{t}_{ik}\alpha\| \leqslant \sqrt{2}k_v & \text{(a)} \\ \sum_{i=1}^{NG} \widehat{\mathbf{B}}_i^{\mathrm{T}} \boldsymbol{\beta}_{ik} = 0 & \text{(b)} \end{cases} \tag{3.7}$$

where: $\|.\|$ *denotes Euclidean vector norm.*

Proof. By setting $\varepsilon^2 = 0$, let us write the Lagrange dual function of (3.6) as:

$$F_{\mathrm{L}} = \sum_{i=1}^{NG} \left\{ \sum_{k=1}^{m} \sqrt{2}k_v \left(\sqrt{\mathbf{e}_{ik}^{\mathrm{T}}\mathbf{e}_{ik}} \right) - \sum_{k=1}^{m} \frac{1}{3} (\boldsymbol{\gamma}_{ik}^{\mathrm{T}} \mathbf{D}_v \mathbf{e}_{ik}) - \boldsymbol{\beta}_i^{\mathrm{T}} \left(\sum_{k=1}^{m} \mathbf{e}_{ik} - \widehat{\mathbf{B}}_i \mathbf{q} \right) \right\} - \alpha \left(\sum_{i=1}^{NG} \sum_{k=1}^{m} \mathbf{e}_{ik}^{\mathrm{T}} \mathbf{t}_{ik} - 1 \right) \tag{3.8}$$

where $\boldsymbol{\gamma}_{ik}$, $\boldsymbol{\beta}_i$, α are Lagrange multipliers. Note that $\boldsymbol{\gamma}_{ik}$, $\boldsymbol{\beta}_i$ are vectors at Gauss point i for each load vertex k while α is merely a scalar.

The dual problem of (3.6) is:

$$\max_{\boldsymbol{\gamma}_{ik},\boldsymbol{\beta}_i,\alpha} (\min_{\mathbf{e}_{ik},\mathbf{q}} F_{\mathrm{L}}) \tag{3.9}$$

Because a finite solution for (3.6) exists, the constraint system (3.6.b–d) is affine and the objective function is convex, then the duality theorem states that there exists no duality gap between primal and dual solutions:

$$\min_{\mathbf{h}(\mathbf{e}_{ik},\mathbf{q})=\mathbf{0}} \sum_{i=1}^{NG} \sum_{k=1}^{m} \sqrt{2}k_v \sqrt{\mathbf{e}_{ik}^{\mathrm{T}}\mathbf{e}_{ik}} = \max_{\boldsymbol{\gamma}_{ik},\boldsymbol{\beta}_i,\alpha} (\min_{\mathbf{e}_{ik},\mathbf{q}} F_{\mathrm{L}}) \tag{3.10}$$

where $\mathbf{h}(\mathbf{e}_{ik}, \mathbf{q}) = \mathbf{0}$ stands for linear constraint system (3.6.b–d).

The Lagrange function (3.8) may be written in another form:

$$F_{\mathrm{L}} = \sum_{i=1}^{NG} \sum_{k=1}^{m} \left(\frac{\sqrt{2}k_v \mathbf{e}_{ik}}{\sqrt{\mathbf{e}_{ik}^{\mathrm{T}}\mathbf{e}_{ik}}} - \boldsymbol{\gamma}_{ik} - \boldsymbol{\beta}_i - \mathbf{t}_{ik}\alpha \right)^{\mathrm{T}} \mathbf{e}_{ik} + \sum_{i=1}^{NG} \boldsymbol{\beta}_i^{\mathrm{T}} \widehat{\mathbf{B}}_i \mathbf{q} + \alpha \tag{3.11}$$

In formulation (3.11) we adopt the convention that if the vector norm of the strain rate $\|\mathbf{e}_{ik}\|$ is equal to zero then: $\frac{\sqrt{2}k_v \mathbf{e}_{ik}^{\mathrm{T}}}{\sqrt{\mathbf{e}_{ik}^{\mathrm{T}}\mathbf{e}_{ik}}} \mathbf{e}_{ik} = 0$.

Due to the existence of a dual solution α^- with zero duality gap, it is required that for any solution set of Lagrange multipliers $(\boldsymbol{\gamma}_{ik}, \boldsymbol{\beta}_i, \alpha)$ the function $\min_{\mathbf{e}_{ik},\mathbf{q}} F_{\mathrm{L}}$ must have a finite value. To this end, the following system must be satisfied:

$$\begin{cases} \left(\frac{\sqrt{2}k_v \mathbf{e}_{ik}}{\sqrt{\mathbf{e}_{ik}^{\mathrm{T}}\mathbf{e}_{ik}}} - \boldsymbol{\gamma}_{ik} - \boldsymbol{\beta}_i - \mathbf{t}_{ik}\alpha \right)^{\mathrm{T}} \mathbf{e}_{ik} \geqslant 0 & \forall \mathbf{e}_{ik} \quad \text{(a)} \\ \sum_{i=1}^{NG} \boldsymbol{\beta}_i^{\mathrm{T}} \widehat{\mathbf{B}}_i \mathbf{q} = 0 & \forall \mathbf{q} \quad \text{(b)} \end{cases} \tag{3.12}$$

otherwise we always have:

$$\min_{\mathbf{e}_{ik},\mathbf{q}} F_{\mathrm{L}} \to -\infty \tag{3.13}$$

According to (3.13), the function of $\min_{\mathbf{e}_{ik},\mathbf{q}} F_L$ is bounded from below:

$$\min_{\mathbf{e}_{ik},\mathbf{q}} F_L \geqslant \alpha \tag{3.14}$$

It reaches α when, for example, all strain rates and displacements are equal to zero. This fact leads to the conclusion:

$$\min_{\mathbf{e}_{ik},\mathbf{q}} F_L = \alpha \tag{3.15}$$

The condition (3.12.b) is equivalent to:

$$\sum_{i=1}^{NG} \mathbf{B}_i^T \boldsymbol{\beta}_i = \mathbf{0} \tag{3.16}$$

We can also point out that the condition (3.12.a) is equivalent to the restriction on only multipliers $\boldsymbol{\gamma}_{ik}$, $\boldsymbol{\beta}_i$, α:

$$\|\boldsymbol{\gamma}_{ik} + \boldsymbol{\beta}_i + \mathbf{t}_{ik}\alpha\| \leqslant \sqrt{2}k_v \quad \forall i, k \tag{3.17}$$

Equalities (3.15), (3.16) and inequality (3.17) conclude our proof. □

Because the formulations (3.6) and (3.7) are primal and dual problems, it is also useful to present this primal–dual forms as a set of stationary conditions as follows:

$$\begin{aligned}
&\frac{\sqrt{2}k_v \mathbf{e}_{ik}}{\sqrt{\mathbf{e}_{ik}^T \mathbf{e}_{ik}}} - (\boldsymbol{\gamma}_{ik} + \boldsymbol{\beta}_i + \alpha \mathbf{t}_{ik}) = \mathbf{0} && \text{(a)} \\
&\mathbf{D}_v \mathbf{e}_{ik} = \mathbf{0} && \text{(b)} \\
&\sum_{k=1}^{m} \mathbf{e}_{ik} - \hat{\mathbf{B}}_i \mathbf{q} = \mathbf{0} && \text{(c)} \\
&\sum_{i=1}^{NG} \hat{\mathbf{B}}_i^T \boldsymbol{\beta}_i = \mathbf{0} && \text{(d)} \\
&\sum_{i=1}^{NG} \sum_{k=1}^{m} \mathbf{e}_{ik}^T \mathbf{t}_{ik} - 1 = \mathbf{0} && \text{(e)}
\end{aligned} \tag{3.18}$$

Further more, it is possible to show that the Lagrange multiplier $\boldsymbol{\gamma}_{ik}$ in the formulation (3.7) can be eliminated and we have:

$$\begin{aligned}
&\alpha^- = \max_{\boldsymbol{\beta}_i, \alpha} \alpha && \text{(a)} \\
&\text{s.t.} \begin{cases} f(\mathbf{D}^{1/2}\boldsymbol{\beta}_i + \alpha \boldsymbol{\sigma}_{ij}^E) \leqslant 0 & \text{(b)} \\ \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \mathbf{D}^{1/2} \boldsymbol{\beta}_i = \mathbf{0} & \text{(c)} \end{cases}
\end{aligned} \tag{3.19}$$

Obviously (3.19) is a discretized form of the shakedown load multiplier formulated by the Melan theorem. In formulation (3.19) the vector $\mathbf{D}^{1/2}\boldsymbol{\beta}_i$ can be interpreted as the stress vector of a time-independent residual stress field: the value of this vector is calculated at each Gauss point, independently of load vertices or, in other word, independently of time.

Proposition 2. *If there exists a finite solution α^+ for the kinematic shakedown load multiplier* (3.6) *with $\varepsilon^2 = 0$ and if the incompressibility is automatically satisfied, then the kinematic formulation has its dual form as the static one formulated by the Melan theorem:*

$$\min_{\mathbf{h}(\mathbf{e}_{ik},\mathbf{q})=\mathbf{0}} \sum_{i=1}^{NG} \sum_{k=1}^{m} \sqrt{2}k_v \sqrt{\mathbf{e}_{ik}^T \mathbf{e}_{ik}} = \max_{\begin{cases} \mathbf{B}^T \bar{\boldsymbol{\rho}} = \mathbf{0} \\ f(\alpha \bar{\boldsymbol{\sigma}}_{ik}^E + \bar{\boldsymbol{\rho}}_{ik}) \leqslant 0 \end{cases}} \alpha \tag{3.20}$$

where f is the von Mises yield function.

This proposition shows that the shakedown load multipliers formulated by static (α^-) and kinematic (α^+) theorems are actually the same. As kinematically admissible finite elements are adopted in the present work, both solutions of α^+ and α^- may represent generally an upper bound approximation of the exact solution, although this is not in the strict sense.

Based on the above discussion on duality, a dual algorithm has been developed. The objective is to obtain simultaneously both primal and dual values by solving the system of stationary conditions (3.18). Unfortunately, solving directly this system is a difficult task, because it results in a system of equations much bigger than that in the case of elastic computation. The resulted system thus requires large amount of computer memory as well as computational effort to solve. In order to keep our problem size as small as possible, we use penalty method to handle the incompressibility and compatibility conditions (3.18.b and c) and use Lagrange multipliers as intermediate variables. For the details of the developed algorithm, we invite readers to refer to Vu et al. (2001, submitted for publication).

As noted before, limit analysis may be considered as a special case of shakedown when the structure is loaded with only one monotonic load, i.e. $[\mu_0, \mu_0]P_0$. The developed algorithm is thus expected to give accurate solutions in limit analysis. Numerical examples presented hereafter shows that such requirement is fairly satisfied.

4. Numerical examples

In our first example, a square plate with a central circular hole is examined. The plate is subjected to two loads p_1 and p_2 varying independently. Plane stress state is considered and von Mises yield criterion is used. Due to the symmetric property of the plate, only one fourth of the plate is modeled by quadrilateral 8-node elements. Firstly, numerical investigation carried out in the case of $R/L = 0.5$, with different meshes (Fig. 1b). The results show that when the mesh is refined, both α^+ and α^- decrease and tend toward a convergent solution. While the obtained α^+ represents a real upper bound using a FE mesh with finite DOFs, the corresponding α^- is obviously not a real lower bound. This is due to the fact that we are using a kinematic

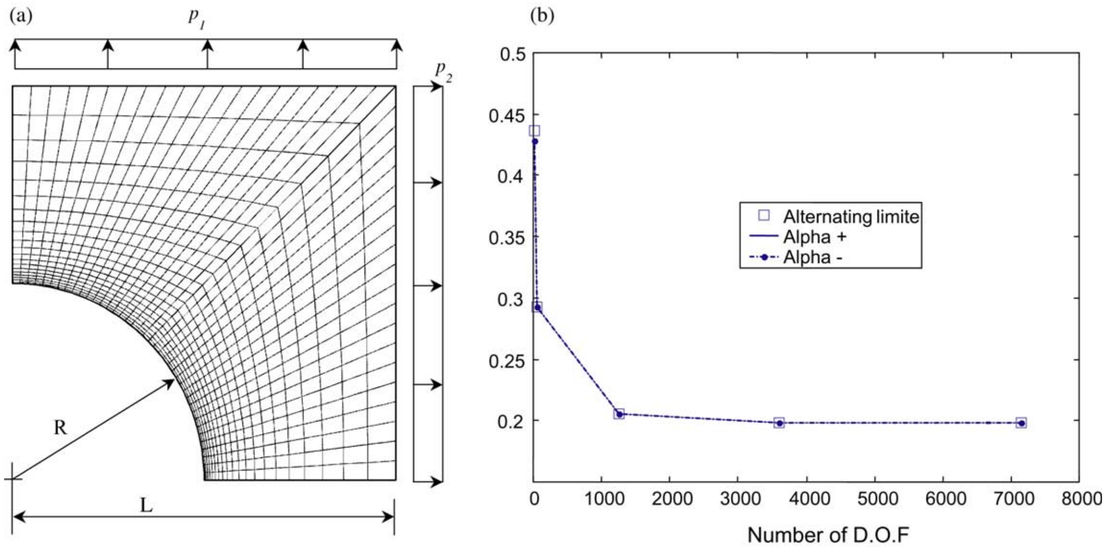

Fig. 1. Plate with a central hole subjected to tension: (a) FEM model and (b) shakedown load multipliers vs. number of DOFs.

finite element mesh (but a static one). It has been shown by the dual theorems presented in Section 3 that α^+ and α^- have a same converge solution. They converge to the exact solution only with appropriate finite element mesh.

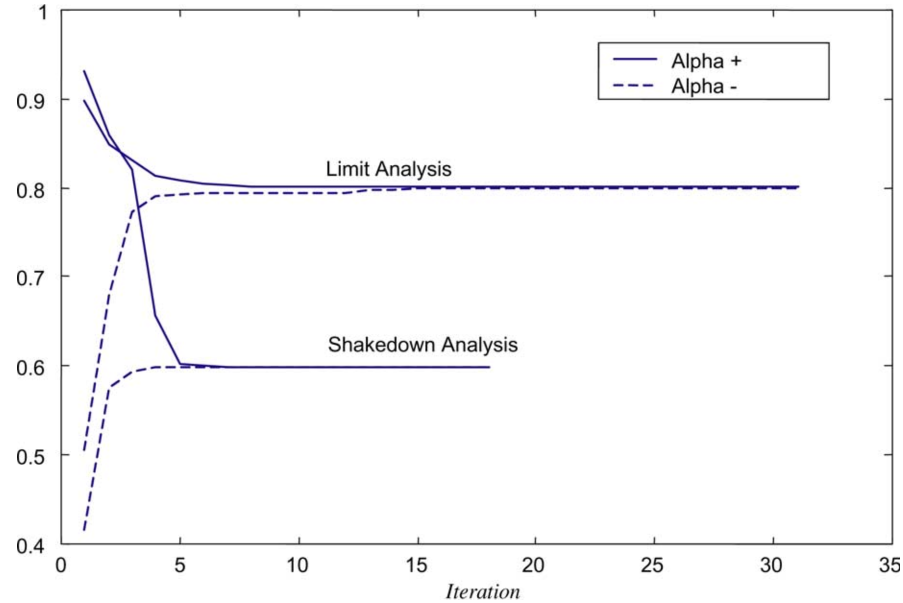

Fig. 2. Limit and shakedown analyses ($R/L = 0.2$, $p_1 \neq 0$, $p_2 = 0$).

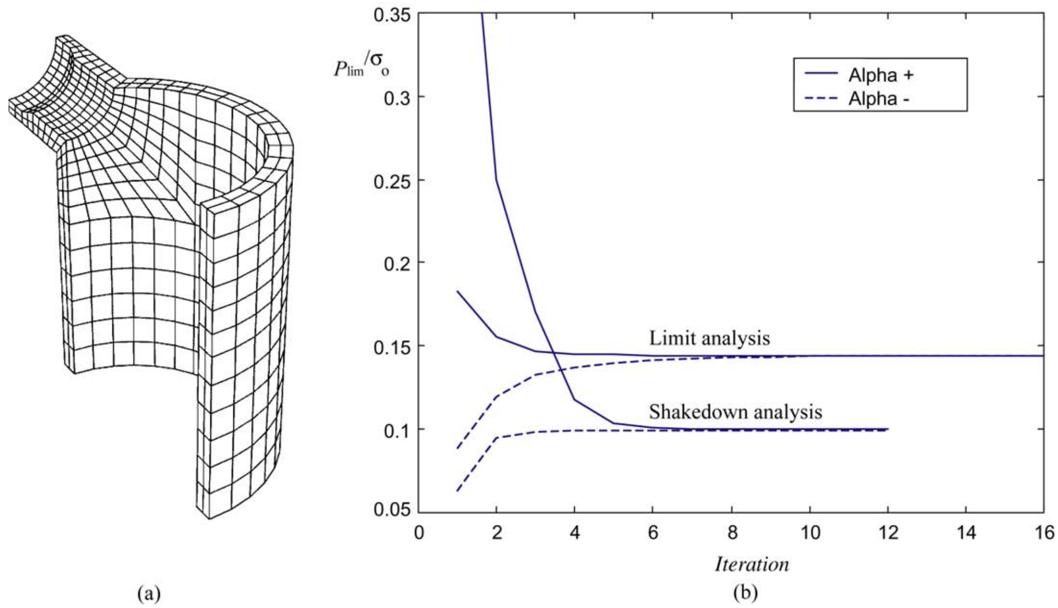

Fig. 3. Pipe junction: (a) FEM mesh and (b) limit and shakedown analysis.

The analytical solution of limit load is known to be exact for $p_1 \neq 0,\ p_2 = 0$ in the case of $0 < R/L \leqslant 0.204$, since in this range the lower bound and upper bound coincide: $\alpha = p_{\lim}/\sigma_0 = (1 - R/L)$ where σ_0 is the yield stress, Gaydon and McCrum (1954). Taking $R/L = 0.2$ as example, the exact limit load multiplier is $\alpha = 0.8$. Our corresponding numerical solutions obtained with 800 elements are $\alpha^- = 0.79924$ and $\alpha^+ = 0.80038$. In shakedown analysis, the shakedown load multiplier for alternating plasticity limit may be estimated as $\alpha_{alter} = p_{\lim}/\sigma_0 = 0.59947$ based on an elastic calculation. Our dual algorithm gives $\alpha^- = 0.59947$ and $\alpha^+ = 0.59949$. Only few iterations are required to obtain the solutions (Fig. 2), showing excellent precision and efficiency of the developed algorithm.

In the second example, we consider a pipe junction subjected to internal pressure p varying within the range $[0, p_0]$. The problem was examined by Staat and Heitzer (1997) who used 125 solid 27-node hexahedron elements for this pipe junction. In our analysis, only one fourth of the structure is modeled because of its symmetric property. The FE mesh presented in Fig. 3(a) contains 720 solid 20-node hexahedron elements. The limit and shakedown analysis results are depicted in Fig. 3(b) where the rapid convergence of solutions may be observed.

5. Conclusions

Using kinematically admissible finite elements and von Mises yield criterion, the duality presented in this paper shows that shakedown load multipliers obtained by Melan and Koiter theorems have the same convergence value which has no strict bounding property. However, in many practical calculations, the convergence solutions may represent generally an upper bound approximation of the exact solution as we are using kinematically admissible finite elements. On the other hand, during the iteration process with a given FE mesh, the solutions (α^+, α^-) remain their bound characteristic with respect to the final convergence solution. Based on the obtained dual formulations of shakedown load multiplier, a new dual algorithm for shakedown analysis of structures is developed. Numerical examples show high calculating efficiency of the algorithm: both primal (α^+) and dual (α^-) values converge rapidly to the accurate solution when an appropriate FE mesh is used. An other paper presented by the same authors (Vu et al., submitted for publication) will discuss mainly the calculating technique of shakedown analysis and will describe the details of the developed algorithm.

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